Target: Precalculus (grades 11 or 12)

Lesson Overview:
In this lesson, students will use their previous knowledge of trigonometry and vector representations of motion to perform the operations of addition and subtraction on given sets of vectors. Students will represent the paths of boats and planes using vectors and investigate the effects of current and wind on these paths using vector addition (i.e. find the altered path). They will also determine a path necessary to keep them on an intended path given the constraints of the current/wind using subtraction of vectors. We've used navigation applications in the past to teach vector operations, but this lesson will hopefully generate more enthusiasm by letting them learn about and actually use a nautical chart as part of the learning process. This lesson should take about two or three 80-minute class periods, given time for a warm-up review of the trig and vectors, introduction to new concepts, learning how to read and use the nautical charts and investigation of vector operations through the activities.

Sources:
- Lessons and practical exercises from the Mathematics & Navigation Teacher Institute at Michigan Tech University, summer 2011
  
  My inspiration for this lesson came from one of our practical exercises where we had to give the captain of our boat the magnetic heading, speed and amount of time to travel to make the section of the trip we’d mapped out. A few minutes into the trip, he realized he was off on the heading by about 4 degrees. This made me think of the vector addition and subtraction we’d done in Precalculus and how we could have used that to determine the new heading and speed we’d need to go to end up where we’d intended.

Materials Needed:
- parallel or rolling rule (one per group)
- laminated copies of NOAA RNC nautical charts #14971 of the Keweenaw Bay and #14972 of Keweenaw Waterway. These maps can be viewed online by going to the following site: http://www.charts.noaa.gov/OnLineViewer/14972.shtml
- vis-à-vis markers (one 4-pack per group)
- calculator
- paper/pencil
- student handout
- dividers or ruler
New Vocabulary:
- true north – refers to the geographic north pole
- magnetic north – tends to shift and refers to the pole of the Earth’s magnetic field
- declination – the difference between true north and magnetic north
- compass rose – two concentric circles, each broken into 360°, where the outer circle measures true directions and the inner circle measures magnetic directions
- bearing – the direction from one object to another (e.g. from you to a fixed object)
- heading – the direction one is pointed at the moment
- parallel rule – a tool used to draw parallel lines. It consists of two unmarked rulers connected by hinges that keep them parallel at all times.
- nautical mile (nm) – the equivalent of approximately 1 minute of latitude. 1 nm ≈ 6076 ft or 1.15 mi
- knot (kn) – a rate or speed representing nautical miles per hour
- latitude – the angular distance between the Equator and points north or south of it on the surface of the Earth.
- longitude – the angular distance between the Prime Meridian and points east or west of it on the surface of the Earth.
- resultant vector – a vector produced by adding and/or subtracting two or more given vectors

Focus Question(s):
After reviewing vector representations of boat and flight paths, I will introduce them to a boating situation where they start at a given position, travel at a certain speed in a certain direction for a given amount of time and then change course, traveling at another speed in a different direction and amount of time. I will then ask: “How could you have gotten to your ending position using one straight-shot path from where you originally started?” I will ask them to represent the situation using a labeled diagram that shows the original “legs” of their tour, along with the “straight-shot” path. This will lead into the concept of adding vectors. They will have only worked with vectors in terms of direction, so we will discuss heading and bearing in terms of navigation. After we’ve investigated adding components of the vectors when two or more changes are made by the boater, I’ll ask them what would happen in the event of a current (or wind for a plane). We’ll then discuss how you can determine the heading and speed you’d need to go in order to maintain an intended path under the constraints of the current/wind, which will lead into subtraction of vectors.

Learning Objectives:
For this lesson, students will have had the following prerequisite knowledge:
- measure angles along the Unit Circle in terms of direction (from the positive x-axis as 0° and moving counterclockwise)
- represent a force or the path of an object, given its magnitude and direction, with a vector.
- break a vector down into its horizontal and vertical components (which can be written in rectangular coordinate form) using a given magnitude and direction and the formulas \( x = r \cos \theta \) and \( y = r \sin \theta \), where \( r \) is the magnitude and \( \theta \) is the direction.
- calculate the magnitude and direction of a vector given its rectangular coordinates using the Pythagorean theorem to calculate the magnitude and arctan to calculate the direction (paying attention to the quadrant in which the vector must lie and making any necessary adjustments to the direction angle)
Once we have reviewed this information, the students will, by the end of the lesson, be able to:

- convert between direction and heading/bearing
- represent the sum or difference of two vectors, along with the resultant vector, using a diagram (Note: we sketch them using both the head-to-tail and tail-to-tail methods, but we will be sketching head-to-tail for the navigation-type problems, as shown in my examples, because the kids find it easier to visualize the situation when sketched in this manner.)
- find the sum and/or difference of two vectors by performing calculations on their horizontal and vertical components to find the components of the resultant vector and, thusly, its magnitude and direction

State of Michigan and National Common Core benchmarks for Precalculus addressed:

- P6.5 Solve trigonometric equations using basic identities and inverse trigonometric functions.
- P7.1 Perform operations (addition, subtraction, and multiplication by scalars) on vectors in the plane. Solve applied problems using vectors.
- N-VM 1, 2 & 3 (common core) Represent and model with vector quantities; properties of vectors, notation, components, applications, etc.

Classroom Activities:

1. Review prerequisite concepts through a warm-up or guided discussion. You do not need to use applied situations, only a review of the basics. Then discuss aspects of boat navigation. Discuss the difference between direction and heading. Have them make sketches of vectors given a direction and have them convert the direction into heading and vice-versa. Show them a nautical chart, the kind of information it displays and how to identify points using latitude and longitude, as well as take measurements using the compass rose and the conversion charts along the edges when measuring distances. Discuss nautical miles and knots and how to convert between nautical miles and feet or statute miles. You'll need to briefly discuss the difference between true north and magnetic north and how we will be using magnetic headings/bearings throughout the lesson. Note: when we move to adding/subtracting vectors, we will be sketching vector diagrams (which will no longer be on the chart itself, but on their own paper) with their horizontal/vertical components in reference to what may appear to be true north in their diagram. However, given that all of our headings/bearings will be magnetic, this will not be an issue due to the relative angles being the same (just rotated).

2. Using either the Keweenaw Waterway or Keweenaw Bay nautical chart, put them into groups of 3 or 4 and have them sketch a vector using a vis-à-vis marker between two points given their lat-long positions. Have them measure the magnitude (in feet) and magnetic heading of the vector using the dividers or a ruler and the compass rose. Review conversions by having them convert the distances in feet to nautical miles, as well as giving them times and asking them to identify the speed in knots they'd need to go in order to get from one place to another in the given amount of time (and vice-versa given speed and calculating time). Next, give them the lat-long of a third location and have them sketch a vector from the second to this third position. Again, have them measure the heading and magnitude (converted also to nautical miles). Discuss how you could get from the first point directly to the third (obviously, with a directed vector) and have them sketch that vector, along with measuring its heading and magnitude.
3. Once they have the three vectors, discuss the concept of adding two vectors to produce a third (resultant) vector. On their own paper (their class notes for the day), have them sketch the three vectors from their nautical chart as shown to the right (Figure 1). Note: I am using a given set of headings and magnitudes for two “legs” of a sample trip and will use these to illustrate the process of adding them and finding the heading/magnitude of the beginning to end trip.

Next, they will make another copy of the diagram, this time adding in the individual components of the first two vectors, as they have done with single vectors in previous activities. (Figure 2) They are essentially creating a right triangle for each vector, with the vector being the hypotenuse and the horizontal leg of the triangle originating from the beginning point of the vector (the tail) - this is VERY important, as they will be converting their heading into direction, which must be measured from the positive x-axis counterclockwise. Notice they have converted their angles to direction (in this case, by subtracting the given headings from 90 degrees - this will not always be the case, so they need to pay attention to the position of the angle.) They have also labeled the horizontal and vertical components for each using lower case letters and subscripts to denote the vectors to which they belong. The capital H and V denote the horizontal and vertical components of the resultant vector (the ones we will be calculating).

Now, they will calculate the lengths of the components using the trig they've already learned. The values are shown to the right. After analyzing the diagram, they should recognize that the horizontal component of the resultant vector, \( H \), is the sum of the two horizontal components and the vertical component of the resultant vector, \( V \), is the sum of the two vertical components. Therefore, they only need to add the two component values together to get the corresponding component values of the resultant, as show to the right.

Once they've calculated the components of the resultant vector, they can apply the Pythagorean Theorem to find its magnitude and the trig to find the direction, which may or may not need to be adjusted depending on the quadrant in which the vector lies. In this case, adjustment is not necessary.

The final step is to convert the direction, which is measured counterclockwise from the positive x-axis, back to heading. In this case, the vector lies in the first quadrant, so we need to find the complement to the angle by subtracting it from 90 degrees.

Once they've seen the example, have them test it out on the set of vectors they were given when working with the nautical charts.
4. Once they’ve seen the process of adding vectors using the situation of a boat controlling its own motion, you can introduce the concept of current (and wind, if you involve planes too). Use examples where the vectors have components that are directed down or to the left (i.e. the values for \( \cos \theta \) and/or \( \sin \theta \) would be negative if using values of \( \theta \) that are greater than 90°). When you draw the vectors and see that the component of one points one way and the corresponding component from the other points in the opposite direction, the resultant component would come from subtracting the absolute valuedistances. This is a good reminder to them that using the actual direction angle (not necessarily the one that lies within the right triangle), will account for subtraction by giving a negative value and they can always add the components.

5. After some practice adding vectors, introduce them to a situation where you have a course you want to take on a given heading and speed and ask what the effects of a current would be given its heading and speed. Ask them what path they would need to take so that the current would “push” them along the intended path and see what they come up with. You may want to have them sketch the intended path, along with the path of the current (as if they were the vectors being added). Then have them think “backwards” as to where a boat would be moving and end up “pushed” to the intended path. They should see that sketching the current vector in the opposite direction will lead them to this path and you would then be subtracting these component values instead of adding them.

6. Assign the attached activity, Vector Madness!, as an in-class assignment or make it homework, or a combination of both. You can assign part of it one day and have them finish the remainder at the conclusion of the entire lesson.

Assessments:

- The conversions between direction and heading can be assessed easily by a spot-check or as part of a warm-up exercise on the second day (or third day, if it goes into one).
- Students will receive a group grade for their work on the activity with the maps (number 2 described above). You should look for correct placement of the vectors from their lat-long positions, reasonable estimates for their headings and magnitudes, and correct conversions to nautical miles.
- You can collect and grade their individual work on the assigned activity, or you can have them present their work in groups for assigned problems. I find this works very well and it gets the kids more involved than me just putting up answers on the board and doing all of the work myself in answering any questions they have. It also allows me to discuss more fully the kind of work I need to see and for them to verify that they have indeed shown that work. I sometimes grade their presentations, focusing more on the work and methods than correct final answers. If you choose to have them present in groups, regardless of whether you assign group grades, you can still collect their individual papers at the end to see what, if any, corrections they were making or if they just sat there and nodded the whole time. I like to do this occasionally to keep them on their toes!
Vector Madness!

1. Given each vector in terms of its heading or direction, make a labeled sketch of the vector and give its equivalent direction or heading.
   a. 100° heading  
   b. 240° direction  
   c. 75° heading  
   d. 115° direction

2. A boat leaves port on a heading of 40° with the automatic pilot set for 12 knots. On this particular day, there is a 6-knot current with a heading of 75°.
   a. Sketch and label vectors to represent the intended path of the boat, the current and the resultant path of the boat with the effects of the current.
   b. Calculate the speed and heading at which the boat will actually travel due to the effects of the current. Show and organize all work!

3. An airplane is traveling on a heading of 170° and its speed in still air is 500 mph. There is a wind blowing at 50 mph on a heading of 65°.
   a. Make and label a vector diagram that shows the intended path of the airplane, the effect of the wind and the altered path of the airplane due to the wind.
   b. Find the speed and heading of the airplane on its altered path. Show and organize all work!

4. The map to the right shows the location of the docked New Hopes when its skipper decided to navigate to Dunes Beach.
   a. What is the heading of the route that the skipper should take to Dunes Beach?
   b. The length of the indicated route is 6.6nm. The maximum speed of the New Hopes is 2.4 knots. At this rate in still water with no wind, how long will it take for the boat to arrive at Dunes Beach?
   c. Suppose that as the New Hopes heads for Dunes Beach, a strong wind moves the boat at 0.8 knot at a heading of 200°. The skipper sticks to the original heading from Part a despite the wind. Make a labeled sketch of the situation that shows the intended boat path, the effects of the wind and the altered path of the boat.
   d. Determine the speed and heading of the boat on its altered path. Show your work!!!
   e. The skipper wants to be moving on course toward Dunes Beach at 2.4 knots, using the wind to help him. What heading and speed should he set to account for the wind and arrive at Dunes Beach as planned? Show your work!!!

5. An airplane is traveling at a speed of 600 mph on a heading of 345°. There is a wind moving at a speed of 70 mph on a heading of 20°. The pilot would like to maintain his planned course. Find the speed and heading he should set in order to travel with the wind and keep on course. Make labeled diagrams and show/organize your work!!!
Answer Key to Vector Madness!

1. a) Direction: \(-10^\circ\) or \(350^\circ\)
   Heading: \(210^\circ\)
 b) Direction: \(15^\circ\)
   Heading: \(335^\circ\)

2. a) b) Speed: about 17.26 knots
    Heading: about \(51.5^\circ\)

3. b) Speed: about 489.45 mph
    Heading: about \(164.34^\circ\)

4. a) heading: \(250^\circ\)
 b) 2.75 hours
 c) d) altered speed: about 2.98 knots
    altered heading: about \(238.12^\circ\)
 e) speed: about 1.98 knots
    heading: about \(268^\circ\)

5. Speed: about 544.14 mph
    Heading: about \(340.77^\circ\)